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Best Approximation in $L^p(I, X)$, $0<p<1$

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Let X be a real Banach space and (Ω, μ) a finite measure space. If ϕ is an increasing subadditive continuous function on $[0, \infty)$ with $\phi(0)=0$, then we set $L^\phi(\mu, X)=\{f: \Omega \to X: \|f\|_\phi = \int \phi(\|f(t)\|)\, d\mu(t) < \infty\}$. One of the main results of this paper is: "For a closed subspace Y of X, $L^\phi(\mu, Y)$ is proximinal in $L^\phi(\mu, X)$ if and only if $L^1(\mu, Y)$ is proximinal in $L^1(\mu, X)$." Hence if Y is a reflexive subspace of X, then $L^p(\mu, Y)$ is proximinal in $L^p(\mu, X)$ for all $0<p<1$. Other results on proximinality of subsets of l^ϕ and $L^\phi(\mu)$ are presented as well.

INTRODUCTION

Let X be a real Banach space and Y a closed subspace of X. For $x \in X$, we let $d(x, Y) = \inf\{\|x-y\| : y \in Y\}$. The subspace Y is called proximinal in X if, for every $x \in X$, there exists $y \in Y$ such that $d(x, Y) = \|x-y\|$. Such an element $y \in Y$ is called a best approximant of x in Y. Set $P(x, Y) = \{y: d(x, Y) = \|x-y\|\}$. In general $P(x, Y)$ is empty. It is a very general and important question "whether a subspace Y is proximinal in X or not." A compactness argument shows that every finite-dimensional subspace Y is proximinal in X. In case X is a metric linear space, then this is no longer true [1]. We refer the reader to Singer [5], for more on best approximation in Banach and metric spaces.

In this paper we study proximinality of some subsets and subspaces of the sequence metric linear space $l^\phi(X)$ and the function metric linear space $L^\phi(X)$, for modulus function ϕ and some Banach space X.

In Section 2, we prove that if Y is any proximinal subspace of X, then $l^\phi(Y)$ is proximinal in $l^\phi(X)$. Further, we prove that if Y is a reflexive subspace of X, then $L^\phi(Y)$ is proximinal in $L^\phi(X)$. In Section 3, the proximinality of some closed subsets in l^ϕ is discussed. Throughout this paper, (Ω, μ) is a finite measure space. $\mathbb{C}$ denotes the complex numbers.

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1. The Spaces $l^\phi(X)$ and $L^\phi(\mu, X)$

A function ϕ: $[0, \infty) \to [0, \infty)$ is called a modulus function if:

(i) ϕ is continuous and increasing,

(ii) $\phi(x) = 0$ if and only if $x = 0$,

(iii) $\phi(x + y) \leqslant \phi(x) + \phi(y)$.

Examples of such functions are $\phi(x) = x^p$, $0 < p \leqslant 1$, and $\phi(x) = \ln(1 + x)$. In fact if ϕ is a modulus function then $\psi(x) = \phi(x)/(1 + \phi(x))$ is a modulus function. Further, the composition of two modulus functions is a modulus function.

For a modulus function ϕ and a measure space (Ω, μ) we set

$$l^\phi = \left\{ (a_n) \colon \sum_{n=1}^{\infty} \phi\, |a_n| < \infty \right\}$$

$$L^\phi(\mu) = \left\{ f \colon \Omega \to \mathbb{C} \colon \int \phi\, |f|\, d\mu < \infty \right\}.$$

For $a = (a_n) \in l^\phi$ and $f \in L^\phi(\mu)$ one can define

$$\|a\|_\phi = \sum_{n=1}^{\infty} \phi\, |a_n| \qquad \text{and} \qquad \|f\|_\phi = \int \phi\, |f|\, d\mu.$$

Then one can easily prove:

LEMMA 1.1. *$(l^\phi, \|\ \|_\phi)$ and $(L^\phi(\mu), \|\ \|_\phi)$ are complete metric linear spaces.*

Further, it is known that $l^\phi \subseteq l^1$ and $L^\phi(\mu) \supseteq L^1(\mu)$. For more on l^ϕ and $L^\phi(\mu)$ we refer the reader to [2–4].

For a Banach space X, we define

$$l^\phi(X) = \left\{ (f(n)) \colon \sum_{n=1}^{\infty} \phi\, \|f(n)\| < \infty, f(n) \in X \right\}$$

$$L^\phi(\mu, X) = \left\{ g \colon \Omega \to X \colon \int \phi\, \|g(t)\|\, d\mu(t) < \infty \right\}.$$

For $f \in l^\phi(X)$ and $g \in L^\phi(\mu, X)$, set

$$\|f\|_\phi = \sum \phi\, \|f(n)\| \qquad \text{and} \qquad \|g\|_\phi = \int \phi\, \|g(t)\|\, d\mu(t).$$

Then the following result is similar to Lemma 1.1:

LEMMA 1.2. *$(l^\phi(X), \|\ \|_\phi)$ and $(L^\phi(\mu, X), \|\ \|_\phi)$ are complete metric linear spaces. If $l^\phi \neq l^1$ $(L^\phi(\mu) \neq L^1(\mu))$, then l^ϕ $(L^\phi(\mu))$ is not locally convex.*

Let $l^\phi \otimes X$ be the algebraic tensor product of l^ϕ with X. Hence

$$l^\phi \otimes X = \left\{ \sum_{i=1}^{n} u_i \otimes x_i \colon u_i \in l^\phi \text{ and } x_i \in X \right\}.$$

For $f \in l^\phi \otimes X$, we define

$$\|f\|_{\pi(\phi)} = \inf \left\{ \sum_{i=1}^{n} \|u_i\|_\phi \cdot \phi \ \|x_i\| \right\},$$

where the infimum is taken over all representations $f = \sum_{i=1}^{n} u_i \otimes x_i$.

LEMMA 1.3. *If ϕ is a modulus function that satisfies $\phi(a \cdot b) \leqslant \phi(a) \cdot \phi(b)$, then $\|\ \|_{\pi(\phi)}$ is a metric on $l^\phi \otimes X$.*

Proof. The only point to be proved is: If $\|f\|_{\pi(\phi)} = 0$ then $f = 0$. To see that:

If $\|f\|_{\pi(\phi)} = 0$, then for every k there exists a representation $f = \sum_{i=1}^{n_k} u_i^k \otimes x_i^k$ such that $\sum_{i=1}^{n_k} \|u_i^k\|_\phi \cdot \phi \ \|x_i^k\| < 1/k$. Hence

$$\sum_{i=1}^{n_k} \left[\phi \ \|x_i\| \cdot \sum_{j=1}^{\infty} \phi \ |u_i^k(j)| \right] < \frac{1}{k}.$$

The subadditivity and submultiplicativity of ϕ imply:

$$\phi \left[\sum_{i=1}^{n_k} \|x_i\| \cdot \sum_{j=1}^{\infty} |u_i^k(j)| \right] < \frac{1}{k}.$$

It follows that $\inf \sum_{i=1}^{n_k} \|u_i^k\|_1 \ \|x_i\| = 0$, where $\|u_i^k\|_1$ is the norm of u_i^k as an element in l^1. It follows that $f = 0$. Q.E.D.

The metric space $l^\phi \otimes X$ need not be complete. We set $l^\phi \hat{\otimes}_\phi X$ to denote its completion. In a very similar way we define $L^\phi \hat{\otimes}_\phi X$. The space $l^\phi \hat{\otimes}_\phi X$ can be considered as a space of ϕ-nuclear operators from c_0 into X.

Using the idea in the proof of Theorem 2.1 [3], one can prove

THEOREM 1.4. *The spaces $(l^\phi \hat{\otimes}_\phi X, \|\ \|_{\pi(\phi)})$ and $(L^\phi \hat{\otimes}_\phi X, \|\ \|_{\pi(\phi)})$ are complete metric linear spaces.*

The following, though simple, is an interesting result:

THEOREM 1.5. *Let ϕ be a submultiplicative modulus function. Then*:

(i) *$l^\phi \hat{\otimes}_\phi X$ is isometrically isomorphic to $l^\phi(X)$,*

(ii) *$L^\phi(\mu) \hat{\otimes}_\phi X$ is isometrically isomorphic to $L^\phi(\mu, X)$.*

Proof. It is enough to prove (ii), for (i) is a special case of (ii). Since simple functions are dense in $L^\phi(\mu)$, it follows that the set of elements of the form $\sum_{i=1}^n 1_{E_i}\ x_i$, $E_i \cap E_j$, is the empty set for $i \neq j$ and is dense in $L^\phi(\mu, X)$ and that the set of elements of the form $\sum_{i=1}^n 1_{E_i \otimes x_i}$ is dense in $L^\phi(\mu) \hat{\otimes}_\phi X$.

For $f \in L^\phi(\mu) \otimes X$, $f = \sum_{i=1}^n u_i \otimes x_i$, the function $F_f(t) = \sum_{i=1}^n u_i(t)\, x_i \in L^\phi(\mu, X)$. Further

$$\|F_f\|_\phi = \sum_{i=1}^n \int \phi\ \|u_i(t)\, x_i\|\ d\mu(t) \qquad \text{(since } \phi \text{ is subadditive)}$$

$$= \sum_{i=1}^n \int \phi\ |u_i(t)| \cdot \phi\ \|x_i\|\ d\mu(t) \qquad \text{(since } \phi \text{ is submultiplicative)}$$

$$\leqslant \sum_{i=1}^n \|u_i\|_\phi \cdot \phi\ \|x_i\|.$$

Since this is true for every representation of f, it follows that $\|F_f\|_\phi \leqslant \|f\|_{\pi(\phi)}$. Thus the map $\mathscr{F}(f) = F_f$ is bounded on a dense subspace of $L^\phi(\mu) \hat{\otimes}_\phi X$.

For $f = \sum_{i=1}^n 1_{E_i} \otimes x_i$, E_i disjoint measurable sets in Ω, one has

$$\|F_f\|_\phi = \sum_{i=1}^n \mu(E_i) \cdot \phi\ \|x_i\|$$

$$= \sum_{i=1}^n \|1_{E_i}\|_\phi \cdot \phi\ \|x_i\|$$

$$\geqslant \|f\|_{\pi(\phi)}.$$

Thus $\mathscr{F}$ is an isometric operator from a dense subspace of $L^\phi(\mu) \hat{\otimes}_\phi X$ onto a dense subspace of $L^\phi(\mu, X)$. Consequently $\mathscr{F}: L^\phi(\mu) \hat{\otimes}_\phi X \to L^\phi(\mu, X)$ is an isometric onto operator. Q.E.D.

2. BEST APPROXIMATION IN $l^\phi(X)$ AND $L^\phi(\mu, X)$

In this section, we assume ϕ is a strictly increasing modulus function and Y a closed subspace of X.

THEOREM 2.1. *If Y is a proximinal subspace of X, then $l^\phi(Y)$ is a proximinal subspace of $l^\phi(X)$.*

Proof. Let $(f(n)) \in l^{\phi}(X)$. Since Y is proximinal in X, for each n, there exists $g(n) \in Y$ such that $d(f(n), Y) = \|f(n) - g(n)\|$. Further, $\|g(n)\| \leqslant 2\,\|f(n)\|$. Consequently $g = (g(n)) \in l^{\phi}(Y)$. We claim that g is a best approximant for f in $l^{\phi}(Y)$. To see that, let h be any element of $l^{\phi}(Y)$. Then

$$\begin{aligned}\|f-h\|_{\phi} &= \sum_{n=1}^{n} \phi\,\|f(n)-h(n)\| \\ &\geqslant \sum_{n=1}^{\infty} \phi\,\|f(n)-g(n)\| \\ &= \|f-g\|_{\phi}.\end{aligned}$$

Hence $d(f, l^{\phi}(Y)) = \|f-g\|_{\phi}$, and $g \in P(f, l^{\phi}(Y))$. Q.E.D.

The subspace Y is called a ϕ-summand of X if there is a bounded projection $Q: X \to Y$ such that $\phi(\|x\|) = \phi\,\|Qx\| + \phi(\|(I-Q)x\|)$ for all $x \in X$ where I is the identity map on X. Clearly every ϕ-summand Y of X is proximinal. In fact for $x \in X$, $Q(x) \in P(x, Y)$. Theorem 2.1 is not true in general for $L^{\phi}(\mu, X)$. However, the following is true:

THEOREM 2.2. *Let Y be a ϕ-summand of X. Then $L^{\phi}(\mu, Y)$ is a 1-summand of $L^{\phi}(\mu, X)$.*

Proof. Let $Q: X \to Y$ be the associated projection for Y. Let $\tilde{Q}: L^{\phi}(\mu, X) \to L^{\phi}(\mu, Y)$ be defined by

$$\tilde{Q}(f)(t) = Q(f(t)).$$

Clearly $\tilde{Q}(f) \in L^{\phi}(\mu, Y)$. Further

$$\phi\,\|f(t)\| = \phi\,\|Q(f(t)\| + \phi\,\|(I-Q)\,f(t)\|.$$

Hence,

$$\int \phi\,\|f(t)\|\,d\mu(t) = \int \phi\,\|Q(f(t))\|\,d\mu(t) + \int \phi\,\|(I-Q)\,f(t)\|\,d\mu(t).$$

So,

$$\|f\|_{\phi} = \|\tilde{Q}(f)\|_{\phi} + \|(I-\tilde{Q})f\|_{\phi},$$

and consequently $\tilde{Q}$ is the required projection. Q.E.D.

As a corollary, we have

COROLLARY 2.3. *If Y is a ϕ-summand of X, then $L^{\phi}(\mu, Y)$ is proximinal in $L^{\phi}(\mu, X)$.*

THEOREM 2.4. *Let Y be a proximinal subspace of X. Then for every simple function $f \in L^\phi(\mu, X)$, $P(f, L^\phi(\mu, Y))$ is not empty.*

Proof. Let $f = \sum_{i=1}^n 1_{E_i} x_i$, where E_i are disjoint measurable sets in Ω. Set $g = \sum_{i=1}^n 1_{E_i} y_i$, where $y_i \in P(x_i, Y)$. If h is any element in $L^\phi(\mu, Y)$, then

$$\begin{aligned}
\|f-h\|_\phi &= \int \phi\, \|f(t)-h(t)\|\, d\mu(t) \\
&= \sum_{i=1}^n \int_{E_i} \phi\, \|f(t)-h(t)\|\, d\mu(t) \\
&= \sum_{i=1}^n \int_{E_i} \phi\, \|x_i-h(t)\|\, d\mu(t) \\
&\geqslant \sum_{i=1}^n \int_{E_i} \phi\, \|x_i-y_i\|\, d\mu(t) \\
&= \int \phi\, \|f(t)-g(t)\|\, d\mu(t).
\end{aligned}$$

Hence $\|f-g\|_\phi = \inf\{\|f-h\|_\phi : h \in L^\phi(\mu, Y)\}$. Q.E.D.

Now we prove the main result of this section:

THEOREM 2.5. *Let Y be a closed subspace of X. The following are equivalent*:

(i) *$L^\phi(\mu, Y)$ is proximinal in $L^\phi(\mu, X)$,*

(ii) *$L^1(\mu, Y)$ is proximinal in $L^1(\mu, X)$.*

Proof. (i) $\to$ (ii). Let $f \in L^1(\mu, X)$. Then $f \in L^\phi(\mu, X)$; hence there exists $g \in L^\phi(\mu, Y)$ such that $\|f-g\|_\phi \leqslant \|f-h\|_\phi$ for all $h \in L^\phi(\mu, Y)$. By an argument similar to the one in Lemma 2.10 of [7] we conclude that $\|f(t)-g(t)\| \leqslant \|f(t)-y\|$ for all $y \in Y$ a.e. t. Since $0 \in Y$ one gets $\|g(t)\| \leqslant 2\,\|f(t)\|$ a.e. t. Hence $g \in L^1(\mu, Y)$. Also $\int \|f(t)-g(t)\|\, d\mu \leqslant \int \|f(t)-\theta(t)\|\, d\mu$ for all $\theta \in L^1(\mu, Y)$.

Conversely, (ii) $\to$ (i). Define a map $J: L^\phi(\mu, X) \to L^1(\mu, X)$ by $J(f) = \hat{f}$ where $\hat{f}(t) = (\phi(\|f(t)\|)/\|f(t)\|)\, f(t)$, if $f(t) \neq 0$ and $\hat{f}(t) = 0$ if $f(t) = 0$. Clearly $\|\hat{f}\|_1 = \|f\|_\phi$. Also since ϕ is one-to-one it follows that J is one-to-one. To show that J is onto, let $g \in L^1(\mu, X)$ and take $f(t) = (\phi^{-1}(\|g(t)\|)/\|g(t)\|)\, g(t)$ if $g(t) = 0$, and zero otherwise. Then $\|f\|_\phi = \|g\|_1$; hence $f \in L^\phi(\mu, X)$ and $J(f) = g$. It is also clear that

$$J(L^\phi(\mu, Y)) = L^1(\mu, Y).$$

Now let $f \in L^\phi(\mu, X)$. Then $\hat{f} \in L^1(\mu, X)$ and there exists $\hat{g} \in L^1(\mu, Y)$ such that $\|\hat{f}-\hat{g}\|_1 \leqslant \|\hat{f}-\hat{h}\|_1$ for all $\hat{h} \in L^1(\mu, Y)$ and the support of $\hat{g} \subseteq$ the support of $\hat{f}$. By Lemma 2.10 in [7],

$$\|\hat{f}(t)-\hat{g}(t)\| \leqslant \|\hat{f}(t)-y\| \qquad \text{for all} \quad y \in Y.$$

Hence

$$\left\| f(t) - \frac{\|f(t)\| \, \phi(\|g(t)\|)}{\|g(t)\| \, \phi(\|f(t)\|)} \, g(t) \right\| \leqslant \left\| f(t) - y \, \frac{\|f(t)\|}{\phi(\|f(t)\|)} \right\|.$$

Now take $h \in L^\phi(\mu, Y)$. Then

$$\frac{\phi(\|f(t)\|}{\|f(t)\|} \, h(t) \in Y \qquad \text{a.e. } t.$$

Hence $\|f(t)-w(t)\| \leqslant \|f(t)-h(t)\|$ a.e. t where

$$w(t) = \frac{\|f(t)\| \, \phi(\|g(t)\|)}{\|g(t)\| \, \phi(\|f(t)\|)} \cdot g(t).$$

Using the fact that $\|g(t)\| \leqslant 2 \, \|f(t)\|$ we will show that $w \in L^\phi(\mu, Y)$ as follows,

$$\begin{aligned} \|w(t)\| &= \frac{\|f(t)\| \, \phi(\|g(t)\|)}{\phi(\|f(t)\|)} \leqslant \frac{\|f(t)\| \, \phi(2 \, \|f(t)\|)}{\phi(\|f(t)\|)} \\ &\leqslant \frac{\|f(t)\| \, [2\phi(\|f(t)\|)]}{\phi(\|f(t)\|)} = 2 \, \|f(t)\|; \end{aligned}$$

hence $w \in L^\phi(\mu, Y)$. Thus $L^\phi(\mu, Y)$ is proximinal in $L^\phi(\mu, X)$. Q.E.D.

In [6], it was shown that if Y is reflexive in X then $L^1(\mu, Y)$ is proximinal in $L^1(\mu, X)$. We now prove that this holds also for $L^\phi(\mu, Y)$ in $L^\phi(\mu, X)$.

COROLLARY 2.6. *If Y is a reflexive subspace of X then $L^\phi(\mu, Y)$ is proximinal in $L^\phi(\mu, X)$.*

Proof. The corollary follows from the above theorem and Theorem 2.2 in [6].

3. Some Proximinal Sets in l^ϕ and $L^\phi(\mu)$

In Banach spaces, there are many conditions that imply the proximinality of a given set. A set $E \subseteq L^\phi(\mu)$ $(E \subset l^\phi)$ is called *pointwise* *compact* if every sequence in E has a subsequence that converges pointwise in E.

THEOREM 3.1. *Let E be a pointwise compact set in $L^\phi(\mu)$. Then E is proximinal.*

Proof. Let $f \in L^\phi(\mu)$, and $f_n \in E$ such that $\|f_n - f\|_\phi \to r = d(f, E) = \inf\{\|f-g\|_\phi: g \in E\}$. Since E is pointwise compact, we can assume with no loss of generality that $f_n(t) \to Z(t)$ a.e. t and $Z \in E$. Thus $\phi\,|f_n(t) - Z(t)| \to_n \phi\,|Z(t) - f(t)|$. Hence, by Fatou's lemma, we get

$$\int \phi\,|f(t) - Z(t)|\,d\mu(t) \leqslant \underline{\lim} \int \phi\,|f_n(t) - f(t)|\,d\mu(t)$$

$$= r.$$

Hence $\|f - Z\|_\phi = r = d(f, E)$. Q.E.D.

As a corollary to Theorem 3.1 we have:

THEOREM 3.2. *Every closed ball $B[x, 1]$ in l^ϕ is proximinal.*

Proof. Let (x_n) be a sequence in $B[x, 1]$. Then (x_n) is a sequence in $l^\perp$, $\|x_n\|_1 \leqslant 1$. Since $l^\perp = c_0^*$, we can assume with no loss of generality that there exists Z, $\|Z\|_1 \leqslant 1$, such that $x_n \to Z$ in the w^*-topology of $l^\perp$. In particular $x_n(k) \to Z(k)$, $k = 1, 2, \ldots$. Thus $\phi\,|x_n(k) - x(k)| \to \phi\,|z(k) - x(k)|$. Using Fatou's lemma we get $\|Z - x\|_\phi \leqslant \underline{\lim}\,\|x_n - x\|_\phi \leqslant 1$. Hence $B[x, 1]$ is pointwise compact in l^ϕ. By Theorem 3.1, $B[x, 1]$ is proximinal. Q.E.D.

THEOREM 3.3. *Let M be a pointwise compact subset of l^1. Then $M_\phi = M \cap l^\phi$ is proximinal in l^ϕ.*

Proof. Let $f \in l^\phi M_\phi$, and $r = d(f, M_\phi)$. Then there exists a sequence $\{f_n\} \subset M$ and a sequence $\{g_n\} \subset B[f, r] = \{k \in l^\phi: h - f\|_\phi \leqslant r\}$, such that $\|f_n - g_n\|_\phi \to 0$. Since $\{f_n\} \subset M$, then there exists a subsequence f_{n_j} which converges coordinatewise to $f \in M$. But $\{f_n - g_n\}$ converges coordinatewise to 0. Hence g_{n_j} converges to f_0 coordinatewise. By Fatou's Lemma we get $\|f_0 - f\|_\phi \leqslant \liminf \|g_{n_j} - f\| \leqslant r$. Hence $f_0 \in B_\phi[f_{0,r}] \cap M_\phi$ so $d(f, M_\phi) = \|f - f_0\|$. Q.E.D.

One might expect that for $x \in l^\phi$, and $d(x, B[0, 1]) = r$, that $B[x, r] \cap B[0, 1]$ contains an extreme point of either $B[0, 1]$ or $B[x, r]$. That this is not the case in general follows from the following example:

EXAMPLE. Let $\phi(x) = x^p$, $p = \frac{1}{2}$, and $l_3^\phi = l_3^p = \{(x_1, x_2, x_3): x_i \in R\}$. Let

$x \in l_3^p$, $\|x\|_p > 1$. Then $x^* = x/\|x\|_p^{1/p} \in B[0, 1]$. Further if $x = \sum_{i=1}^3 x_i e_i$, then

$$\|x - x^*\|_p = \sum \left| x_i - \frac{x_i}{\|x\|_p^{1/p}} \right|^p$$
$$= \frac{1}{\|x\|} \cdot \sum |x_i|^p \cdot [\|x\|_p^{1/p} - 1]^p$$
$$= [\|x\|_p^{1/p} - 1]^p.$$

Now, choose $x = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. For any r, the extreme points of $B[x, r]$ are of the form $(\frac{1}{2}, \frac{1}{2}, a)$, $(\frac{1}{2}, a, \frac{1}{2})$, $(a, \frac{1}{2}, \frac{1}{2})$, where $|a - \frac{1}{2}| \leqslant \sqrt{r}$. Hence for any such extreme point θ, we have

$$\|\theta\| = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \sqrt{|a|} > 1.$$

Thus no extreme point of $B[x, r]$ can be on $B[0, 1]$.

On the other hand,

$$\|x - x^*\|_p = [\|x\|^2 - 1]^{1/2}$$
$$= [\tfrac{9}{2} - 1]^{1/2} = \sqrt{\tfrac{7}{2}}.$$

But

$$\|x - e_i\|_p = \frac{3}{\sqrt{2}} > \sqrt{\frac{7}{2}} = \|x - x^*\|,$$

for all $i = 1, 2, ..., 6$, where (e_i) are the extreme points of $B[0, 1]$. Hence no extreme point of $B[0, 1]$ can be in $B[x, r]$.

We remark that the previous example works also for the space $l_3^{\frac{1}{2}}$. So the distance, even in case of finite-dimensional Banach spaces, need not be attained at extreme points.

Let $B_1 = B_1[0, 1] = \{x \in l^1 \colon \|x\|_1 \leqslant 1\}$. Set $B_{1\phi} = B_1 \cap l^\phi$ and $B_\phi = \{x \in l^\phi\colon \|x\|_\phi \leqslant 1\}$.

Remark. For a modulus function ϕ there exist $a > 0$, $\alpha > 0$ such that $\phi(x) \geqslant \alpha x$ for all x in $[0, a)$ (see [4]).

THEOREM 3.4. *Suppose ϕ is a strictly increasing modulus function such that $\phi(x) \geqslant x$ in $(0, 1)$, $\phi(1) = 1$. Then the closed convex hull of B_ϕ in l^ϕ is $B_{1\phi}$.*

Proof. Let $E_n = \{x \in l^\phi\colon \sum_{i=1}^n |x_i| \leqslant 1\}$. We claim that E_n is closed in l^ϕ.

To show this let $x \notin E_n$. Then $\sum_{i=1}^n |x_i| > 1 + \varepsilon$ for some ε; $0 < \varepsilon < 1$. Now if $y \in B[x, \varepsilon] \cap E_n$, then $\sum_{i=1}^n |y_i| \leqslant 1$ and $\|x-y\|_\phi \leqslant \varepsilon$. But $|x_i| \leqslant |x_i - y_i| + |y_i|$ hence

$$1 + \varepsilon < \sum_{i=1}^n |x_i| \leqslant \sum_{i=1}^n |x_i - y_i| + \sum_{i=1}^n |y_i|$$
$$\leqslant \|x-y\|_1 + 1$$

so $\|x-y\|_1 > \varepsilon$.

But $\sum_{i=1}^n \phi(|x_i - y_i|) < \varepsilon$, so $|x_i - y_i| < \phi^{-1}(\varepsilon) < 1$ for all $i = 1, 2, ..., n$. Hence $\phi(|x_i - y_i| \geqslant |x_i - y_i|)$ so $\varepsilon \geqslant \|x-y\|_\phi \geqslant \|x-y\|_1 > \varepsilon$. This contradiction proves that $B[x, \varepsilon] \cap E_n = \phi$; hence E_n is closed. But clearly $B_1 = \bigcap_{n=1}^\infty E_n$ which proves that $B_{1\phi}$ is closed in l^ϕ.

Let $\overline{\text{co}}\, B_\phi$ be the closed convex hull of B_ϕ in l^ϕ. If $x \in B_\phi$ then $\sum_{i=1}^\infty \phi(|x_i|) \leqslant 1$ so $|x_i| \leqslant 1$ for all i, so $|x_i| \leqslant \phi\, |x_i|$. Hence $\|x\|_1 \leqslant \|x\|_\phi \leqslant 1$ which implies that $B_\phi \subset B_{1\phi}$. But $B_{1\phi}$ is closed and convex so $\overline{\text{co}}\, B_\phi \subset B_{1\phi}$.

Now let $x \in B_{1\phi}$, $x = \sum_{i=1}^\infty x_i e_i = \sum_{i=1}^\infty |x_i|\, e_i^*$ when $e_i^* = e_i$ if $x_i \geqslant 0$ and $e_i^* = -e_i$ if $x_i < 0$. Let $\varepsilon > 0$ be given, choose n such that $\sum_{n+1}^\infty \phi(|x_i|) < \varepsilon$, and let $x^* = \sum_{i=1}^n |x_i|\, e_i^*$; since $\sum_{i=1}^n |x_i| \leqslant 1$, then $x^* \in \text{co}\, B_\phi$, but $\|x - x^*\|_\phi = \sum_{i=n+1}^\infty \phi(|x_i|) < \varepsilon$ and hence $x \in \overline{\text{co}}\, B_\phi$ which proves our theorem. Q.E.D.

COROLLARY 3.5. *$\overline{\text{co}}\, B_\phi$ is proximinal in l^ϕ.*

Proof. The proof follows from Theorems 3.3 and 3.4, and the fact that B_1 is pointwise compact in l^1 as shown in the proof of Theorem 3.2.

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